# SEMI-ANALYTICAL METHODS FOR THE SOLUTION OF FUEL DEPLETION PROBLEM IN A NUCLEAR REACTOR

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NUCLEAR ENGINEERING AND TECHNOLOGY PROGRAMME

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CERTIFICATE

Certified that this work entitled "Semi Analytical Methods for the Solution of Fuel Depletion Problem in a Nuclear Reactor" has been done by Shri Keshab Gangopadhyay under my supervision and has not been submitted elsewhere for a degree.

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# Abstract

Fuel depletion problem in a nuclear reactor is a nonlinear constrained initial value problem. The present study makes an analytic approach for the solution of depletion problem for a slab reactor. The methods adopted here are called semi-analytical in the sense that although closed-form solutions are obtained in the form of series, different coefficients occuring in the series are to be evaluated numerically. Small-time-step solution provides a rough but quick estimate of nuclear parameters - which is very important for preliminary nuclear power plant design. In the variational formulation with spatial power smoothening as desirable criteria, it is shown how the performance index may be constructed and simplified to avoid nonlinearity. A technique for handling non-linear performance index is found out in the minimal control effort formulation of the depletion problem. Galerkin Method of weighted residuals is used to find the relation between the coefficients of expansions of trial functions for the state variables. Study of the depletion problem after refuelling is also done. A method has been indicated to find out the refuelling time for achieving maximum benefit. Finally, some recommendations are made for the improvement of the problem and its solution.

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List of Figures and Tables

# CHAPTER - 1

1.1 An important item in the cost of fuel per kilowatt hour of electricity produced by a nuclear power plant is the fuel cost namely expenses incurred in the processing, the fabrication of new fuel elements and refuelling. Except in the case of reactors like CANDU where continuous fuelling is possible, reactors are normally shut down for refuelling resulting in less utilization. So from economic point of view, burnup of fuel should be as large as permissible with proper limitations set by nuclear parameters namely reactivity and also from mechanical and metallurgical considerations. The effect of irradiation puts a limiting factor to the burnup by causing dimensional change and physical damage to both fuel materials and cladding. When burnup is not restricted by radiation damage, the net reactivity change due to the consumption of fissile material (burnup), the accumulation of fission products and the buildup of isotopes, the capture of neutrons by control materials and other poisons determines the life time of the fuel elements. From the above discussion burnup (fissile material depletion) study is very important for an economic nuclear power plant design.

1.2 Martin Becker[1] solved the depletion problem in a slab nuclear reactor by Least Square Variational technique and showed

that his method gave better results compared to other variational methods namely Galerkin method, collocation method, Ritz method. Dougherty and Shen[2] applied a semi-direct variational method to determine the time dependent coefficients of a modal expansion of the neutron fluxes for the multigroup kinetic equations. Kyong[3] has presented a direct technique for the solution of a class of optimal control problems involving a distributed parameter reactor and a new method for solving simultaneous Fredholm integral equation of the second kind which arisesin the optimal control problem. Koyose and Suzuki[4] have shown by using optimization methods in the fuel management and burnup studies of nuclear power reactors, that the requirement for maximum fuel burnup does not lead to power distribution flattening. In the study of Fuller[5] a trial solution is formed for the neutron flux by making expansions in known spatially dependent functions called trial functions. The undetermined time-dependent functions, called amplitude functions are then found by using the weighted residual procedure. Kessler[6] has used the time discontinuous synthesis method to describe the space-dependent dynamics behaviour of fast reactors. The space dependency of the neutron flux, the material temperature and the feedbacks are treated in cylindrical geometry. Two dimensional trial functions for the multigroup fluxes and the concentrations of the precursors are determined in an iterative procedure. The methods of variational synthesis are applied by Stacey[7] to the problem of computing the

optimal control for spatially dependent reactor models. Furthermore a generalized formalism is developed using one group neutron diffusion theory, with and without delayed neutrons. Terney and Fonech[8] applied dynamic programming and a direct flux synthesis method to determine the optimal control rod programming which minimizes power peaking throughout core life for a pressurized-water reactor (PWR). Motoda and Kawai[9] discussed a three dimensional multigroup diffusion theory code for use in fast reactor analysis. The code can be used for computation of material burnup and fission product buildup for specified time intervals. A variational treatment of the burnup optimization of continuous refuelling is presented and numerical solutions are given for a slab reactor by Motoda[10]. Kalimullah[11] applied synthesis programming to solve some problems of optimal control-rod programming of a nuclear reactor discribed by the point kinetics model. Furthermore, he presented four computer codes to solve different kinetics problems by synthesis method. Natelson[12] discussed in his paper two schemes in which solutions to the three dimensional transport equation can be synthesized from two dimensional transport solutions. Derivations are presented which employ a weighted residual technique applied to the second order form of the transport equation. In Stacey's[13] paper the developments of variational principles that admit discontinuous trial functions which need satisfy neither the final and initial conditions nor the external boundary conditions of the physical problem are reviewed

and generalized. He discussed in his later paper[14], the mathematical difficulties that arise when discontinuous trial functions are substituted for continuous functions. These discontinuous trial functions are investigated by formulating the problem in terms of wellknown step functions and their derivatives.

1.3 By studying different works till now the Least Square Variational method is found to be a powerful technique for solving reactor problems. But this method in general leads to nonlinear equations which are time consuming to solve. Hence in the present work suitable construction of the performance index is made leading to a set of linear algebraic equations. Most of the work in the fuel deplection problem uses maximum burnup as the main objective while attempts are made here with new objectives as spatial power smoothening and minimal control effort. The nonlinearity in the performance index is removed either by proper substitution or the use of linearization technique of the actual nonlinear problem. The construction of trial function is made in such a way that several functions involved in the process are analytically integrable using orthogonal property. Thus numerical integration is avoided and hence computation time is saved.

1.4 The fuel-depletion problem in a slab reactor of thickness 2a is considered for the present study. The time-dependent one-group diffusion equation is given by,

$$\frac{\partial}{\partial x}\{D(x,t)\frac{\partial\cancel{\phi}}{\partial x}\phi(x,t)\} + (\nu\,\sigma_f - \sigma_a)N(x,t)\,\phi(x,t) - \textstyle\sum_{ap}(x,t)\,\phi(x,t) = \frac{1}{V}\frac{\partial\phi}{\partial t}\,(x,t) \qquad (1.1)$$

where,

| | | |
|---|---|---|
| $\phi(x,t)$ | = | thermal neutron flux |
| $D(x,t)$ | = | diffusion coefficient |
| $\nu$ | = | average number of neutrons per fission |
| $\sigma_f$ | = | fission crosssection |
| $\sigma_a$ | = | absorption crosssection |
| $N(x,t)$ | = | atom density |
| $\sum_{ap}(x,t)$ | = | macroscopic absorption crosssection of poison material |
| $V$ | = | speed of thermal neutron. |

The arguments will be omitted in next equations for simplicity.

In fuel-depletion problem, the significant change in flux and power distribution takes place over the course of several months. So, the time derivative in equation (1.1) can be safely omitted since flux variations in small time intervals are negligible. Thus, for practical consideration the above equation (1.1) may be treated as a steady state equation,

$$\frac{\partial}{\partial x}\left(D\frac{\partial\phi}{\partial x}\right) + (\nu\sigma_f - \sigma_a)\,N\,\phi - \textstyle\sum_{ap}\,\phi = 0 \qquad (1.2)$$

The variation of fissile isotope concentration is represented by the following equation

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N \tag{1.3}$$

In the present problem it is assumed that there is only one fissile isotope and there is no buildup of fission products. Also, one group diffusion theory is assumed. Initial flux and fuel concentration are given at the start of the reactor. The depletion problem will be solved with the diffusion theory boundary condition that the neutron flux goes to zero on the boundary of the core.

Finally the constraint of the problem is that the reactor operates at a specified power $P'(t)$. Mathematically, it is given as an integral constraint involving only the space variable,

$$G \sigma_f \int_0^a dx\, N \phi = P'(t)$$

i.e.,

$$\int_0^a dx\, N \phi = P(t) \tag{1.4}$$

where $G$ = energy produced per fission

$P(t) = P'(t)/G \sigma_f$

$P(t)$ is given as a polynomial of degree M-1

$$P(t) = \sum_{m=1}^{M} P_m t^{m-1} \tag{1.5}$$

Generally reactors are designed for constant power operation, but variations upto a maximum of 10% can be allowed. Beyond the 10%

variation a reactor is normally shut down for safety. Thus if $P_1$ is the initial reactor power then the contribution of other terms of the polynomial never exceed 10% of $P_1$.

Summarizing, it can be said that the fuel depletion problem is a nonlinear constrained initial value problem in the form of partial differential equations with an integral constraint.

1.5 Chapter 2 makes an attempt for the solution of the depletion problem in small time steps by the assumption of neutron flux remaining constant over each small time step. Due to this assumption the solution is greatly simplified both derivationwise and computationwise. It will be shown later that the results obtained are fairly accurate compared to existing methods. Chapters 3,4,5 are variational formulations of the depletion problem. The present problem is similar to that taken by Martin Becker in his doctoral thesis[15]. In Becker's least square variational formulation solution of a system of nonlinear algebraic equations is involved, whereas in the optimal control method adopted here the functional can be so constructed that the problem reduces to the solution of a set of linear algebraic equations. Thus the method adopted here is better from the computational point of view. The finite element method may also be adopted to solve the depletion problem. Although the desired accuracy may be achieved, computer time and data management may cause serious problem in the FEM. Another difficulty is that different approximations

in the FEM should be tested to know exactly which of these approximations gives the most accurate result with reasonable computer time.

Chapters 3,4 deal with an optimal control problem desired criteria being spatial power smoothening. In Chapter 4 fuel refuelling studies are made. The effect of refuelling decisions on the operating time and hence on the fuel cost is also given in Chapter 4. Chapter 5 deals with a minimal control effort problem for constant power output. A technique for handling nonlinear performance index is indicated in this chapter.

CHAPTER - 2

In this chapter, the depletion problem is solved numerically in small time steps assuming that neutron flux in each time step remains constant. The governing differential equations and constraint (constant power) equation are rewritten here for easy reference.

$$D \frac{\partial^2 \phi}{\partial x^2} + (\nu \sigma_f - \sigma_a) N \phi - \Sigma_{ap} \phi = 0 \qquad (2.1)$$

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N \qquad (2.2)$$

$$\int_0^a dx \; N\phi = P(t) \qquad (2.3)$$

Symbols have their usual meaning as described in chapter 1. These equations are solved with the following boundary conditions

$$\left.\begin{aligned} \phi(x,0) &= \phi_o(x) = A_o \operatorname{Cos} \frac{\pi x}{2a} \\ \text{and} \quad N(x,0) &= N_o \end{aligned}\right] \qquad (2.4)$$

where $A_o$ and $N_o$ are constants.

Also, neutron flux is zero on the boundary of the core i.e.,

$$\phi(a,t) = 0 \qquad \forall\, t \qquad (2.5)$$

This kind of small time-step-solution is a very common and

simplified approach to solve nonlinear partial differential equations. Although it is not a new technique, such an explicit solution for flux and atom density has not been sofar applied in the study of fuel depletion problems. The method was suggested by Zweifel[16] in his text book on Reactor Physics for the burnup problem.

If the reactor operates at constant flux, $\phi$ is independent of time. However, reactors generally operate at constant power; since N changes with time, then $\phi$ will also change with time. That is, $\phi$ is really a function of N, because as the uranium burns the resultant decrease in macroscopic cross-section will tend to make the flux rise to keep the product constant.

The long-term time-dependent reactor equations are integrated in small time steps, in each of which $\phi$ is assumed independent of t. Then at each step a new N is calculated, the corresponding $\phi$ is calculated and the procedure is repeated. One difficulty is that as the fuel is depleted, the control rods will be withdrawn to maintain a critical condition. This tends to change the flux distribution in the reactor. However, the short-time-step approximation is generally adequate to account for such slow variation of flux with time.

2.2 The fissile number density as given by equation 2.2 is,

$$\frac{\partial N}{N} = - \sigma_a \phi \, dt$$

By integration and using the initial condition,

$$N(x,t_1) = N_o\, e^{-\int_0^{t_1} \sigma_a\, \phi\,(x,t)\, dt} \tag{2.6}$$

If $\phi(x,t)$ is assumed to be constant and equal to its initial flux value for the entire first time step $t_1 = \Delta t$ after the start of the reactor, then $N(x, \Delta t)$ is given by,

$$N(x, \Delta t) = N_o\, e^{-\sigma_a\, \phi_o(x)\Delta t} \tag{2.7}$$

Then from the constant power constraint equation 2.3,

$$\int_0^a dx\ N(x, \Delta t)\ \phi(x, \Delta t) = P(\Delta t) \tag{2.8}$$

Substituting the spatial variation of $\phi(x, \Delta t)$ by a cosine function, which is the solution of the neutron diffusion equation in a bare reactor and the time dependence by an amplitude function the equation 2.8 becomes

$$\int_0^a dx\ A_{(0 \rightarrow \Delta t)}\ \cos\frac{\pi x}{2a}\ N(x, \Delta t) = P(\Delta t) \tag{2.9}$$

where $\phi(x, \Delta t) = A_{(0 \rightarrow \Delta t)} \cos\frac{\pi x}{2a}$

The amplitude $A_{(0 \rightarrow \Delta t)}$ will be a constant over the time step $\Delta t$, but changes in the subsequent time intervals.

For the first time step from the above equation, the amplitude function is given by,

$$A_{(0 \to \Delta t)} = \frac{P(\Delta t)}{\int_0^a dx\, N(x, \Delta t) \cos \frac{\pi x}{2a}} \tag{2.10}$$

The integration in the denominator is carried-out numerically by dividing the reactor into well defined zones and then applying Simpson's Rule. Thus the amplitude term $A_{(0 \to \Delta t)}$ is evaluated for the first time interval.

The flux $\phi(x, \Delta t)$ thus evaluated, is assumed constant for the next interval between $\Delta t$ and $2\Delta t$. Fissile atom density $N(x, 2\Delta t)$ is given by,

$$N(x, 2\Delta t) = N(x, \Delta t)\ e^{-\sigma_a \phi(x, \Delta t)\,.\,\Delta t} \tag{2.11}$$

By repeating the procedure described above, the amplitude term $A_{(\Delta t \to 2\Delta t)}$ and hence $\phi(x, 2\Delta t)$ can be evaluated.

The poison cross-section at each time step is calculated from the neutron balance equation 2.1 as,

$$\Sigma_{ap} = \frac{\partial}{\partial x}\left(D \frac{\partial \phi}{\partial x}\right) / \phi + (\nu\ \sigma_f - \sigma_a)\ N \tag{2.12}$$

The procedure is repeated to calculate $\phi$ and $N$ at each time step until $\Sigma_{ap}$ is reduced to a certain preset value. When

the preset value is reached the reactor cannot operate any longer and hence refuelling becomes necessary. Since the present work was compared with Becker's[15] Least Square variational method where the preset value of $\sum_{ap}$ was not available, the problem was continued for various time intervals upto a maximum of 300 days. The final burnup data was obtained at all time steps upto the maximum of 300 days.

2.3 Results of the depletion problem for a slab reactor of width 200 cm fueled with $U_{235}$ ( $\frac{N_{25}}{N_{28}} = .4$ ) with initial multiplication factor of 1.032 are discussed in this section. To compare the solutions of the present work with existing solutions available in literature, the input parameters used by Becker[15] have been taken for the present problem.

$$D = 1 \text{ cm} \quad \text{(constant)}$$

$$\sigma_a = 200 \text{ barns } (1 \text{ barn} = 10^{-24} \text{ cm}^2)$$

$$\nu\sigma_f = 400 \text{ barns}$$

$$P(t) = 6.9 \times 10^{39} \text{ cm}^{-4} \text{ sec}^{-1}$$

$$\phi(x) = A_o \cos \frac{\pi x}{2a}$$

where, $A_o = 6.284 \times 10^{13} \text{ cm}^{-2} \text{ sec}^{-1}$

$$N_o = 2 \times 10^{19} \text{ cm}^{-3}$$

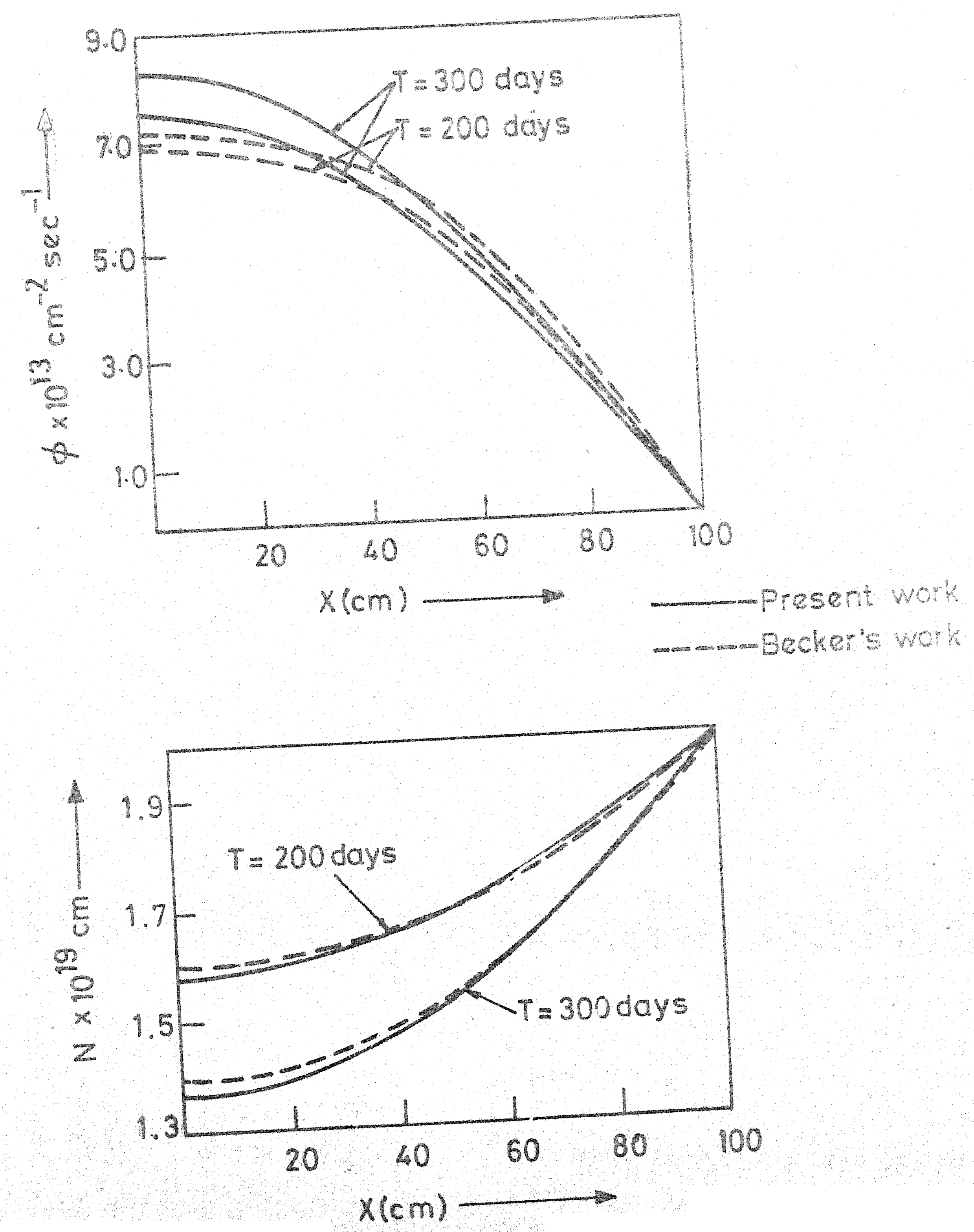
$$\sum_{ap\ t=0} = 3.7605 \quad 10^{-3} \text{ cm}^{-1}$$

$$\Delta t = 0.5 \text{ day}$$

The table 2.1 shows the neutron flux distribution and fuel distribution over space for different reactor operating times for both the present work and Becker's work. These results are plotted in figure 2.1. The figure 2.2 shows the variation of poison cross-section with reactor operating times. Note that the poison cross-section goes down steadily because of the assumption that there is no conversion of $U^{238}$ to $Pu^{239}$ and only $U^{235}$ is the fissile material which is constantly depleted. Becker's results are based on the principle of minimizing the least square error of accurate solution and the approximate **trial function (synthesis) method.** The diffusion theory boundary conditions are applied in the present work and as well as in Becker's analysis. The finite difference results of this chapter are compared with Becker's work only to establish the credibility. Later on the poison cross section variation obtained in this chapter is compared with the cross section obtained based on minimal effort criterion. Although the method adopted here is simplified and approximate, the results are close to Becker's solution within a maximum error of 13%. The discrepancy is due to the assumption of a cosine function for spatial flux variation as opposed to a trial function in the form of a series in Becker's problem and also due to the assumption that flux remains constant in each time step whereas flux is a continuous function of time in Becker's method.

| x cm | $\phi \times 10^{-13}$ T=200 days | | $\phi \times 10^{-13}$ T=300 days | | $N \times 10^{-19}$ T=200 days | | $N \times 10^{-19}$ T=300 days | |
|---|---|---|---|---|---|---|---|---|
| | Present solution | Becker's solution | Present solution | Becker's solution | Present solution | Becker's solution | Present solution | Becker's solution |
| 0 | 7.504 | 6.905 | 8.337 | 7.215 | 1.580 | 1.600 | 1.378 | 1.397 |
| 10 | 7.412 | 6.855 | 8.234 | 7.174 | 1.584 | 1.601 | 1.384 | 1.402 |
| 20 | 7.137 | 6.695 | 7.928 | 7.042 | 1.598 | 1.612 | 1.403 | 1.418 |
| 30 | 6.686 | 6.412 | 7.428 | 6.798 | 1.621 | 1.630 | 1.435 | 1.445 |
| 40 | 6.071 | 6.000 | 6.744 | 6.411 | 1.653 | 1.656 | 1.479 | 1.485 |
| 50 | 5.306 | 5.408 | 5.895 | 5.847 | 1.693 | 1.692 | 1.537 | 1.538 |
| 60 | 4.411 | 4.652 | 4.900 | 5.076 | 1.741 | 1.737 | 1.607 | 1.605 |
| 70 | 3.407 | 3.717 | 3.784 | 4.083 | 1.796 | 1.790 | 1.688 | 1.686 |
| 80 | 2.319 | 2.617 | 2.576 | 2.876 | 1.859 | 1.852 | 1.782 | 1.779 |
| 90 | 1.174 | 1.381 | 1.304 | 1.491 | 1.927 | 1.921 | 1.886 | 1.882 |
| 100 | 0.000 | 0.065 | 0.000 | 0.008 | 1.999 | 1.994 | 1.999 | 1.991 |

Table 2.1 : Fuel & flux distribution for different reactor operating times



T = 300 days
T = 200 days
9.0
7.0
5.0
3.0
1.0
$\phi \times 10^{13}$ cm$^{-2}$ sec$^{-1}$
20
40
60
80
100
X (cm)
Present work
Becker's work
1.9
1.7
1.5
1.3
N $\times 10^{19}$ cm
T = 200 days
T = 300 days
20
40
60
80
100
X (cm)


FIG. 2.1

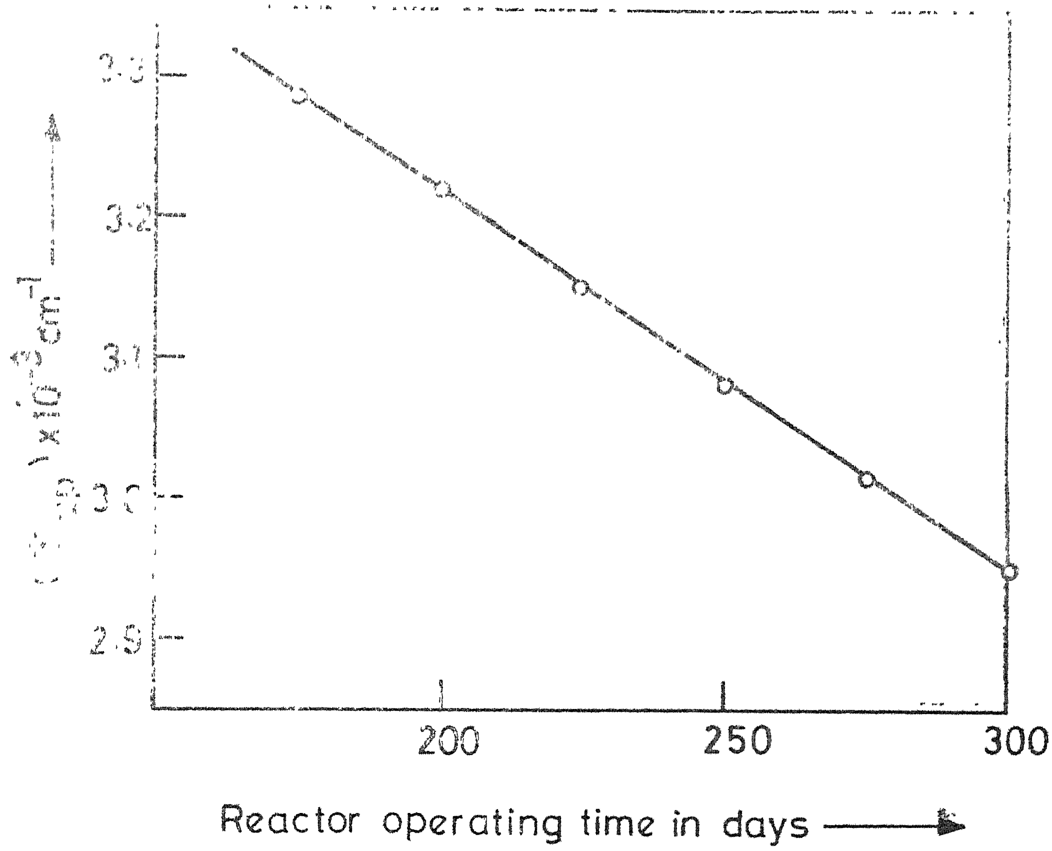



FIG. 2.2

CHAPTER - 3

3.1 In this chapter an attempt is made to attack the depletion problem as an optimal control problem and the solution is achieved by the technique of calculus of variation. The equations concerned are stated again for easy reference.

$$D \frac{\partial^2 \phi}{\partial x^2} + (\nu \sigma_f - \sigma_a) N\phi - \sum_{ap} \phi = 0 \tag{3.1}$$

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N \tag{3.2}$$

$$\int_0^a dx\, N\phi = P(t) \tag{3.3}$$

These equations are to be solved with the same boundary conditions as in chapter 2.

In the language of optimal control theory, N and $\phi$ are called state variables and $\sum_{ap}$ is called the control variable. The objective of the optimal control problem is to achieve smooth power density throughout the space over the entire operating time of the reactor. It is desirable to have smooth power density from the engineering design aspect since the core temperature will be uniform and constant and the total power output will be higher. The quadratic performance index for the present problem can be written mathematically as,

$$P.I = \int_0^a \int_0^T [\alpha_1 (N\phi - \frac{P}{a})^2 + \alpha_2 (\frac{\partial}{\partial x} N\phi)^2]\, dx\, dt \tag{3.4}$$

where $\alpha_1$, $\alpha_2$ = weight factors.

$N\phi$ = power density.

$P(t)/a$ = average power density at time t.

The weighting factors $\alpha_1$ and $\alpha_2$ are arbitrarily chosen to give the appropriate importance either to the variations in the power density or the gradient of power density. This quadratic performance index is to be minimized by proper selection of the state-variables N and $\phi$

Note that the control variable $\sum_{ap}$ does not occur in 3.4 explicitly. The control variable appears only in the governing differential equation 3.1 which is same as diffusion equation. Since the control variable does not appear in the performance index, according to optimal control theory the performance index can be minimized satisfying the dynamics equation 3.2, the integral constraint 3.4 and the initial and boundary conditions on the state variables. Thus by proper choice of the performance index the control variable may be omitted from the main optimal control problem which will make computation easier. When the reduced problem is solved for N and $\phi$ by the calculus of variation technique then the control variable $\sum_{ap}$ is found out easily from the neutron balance equation 3.1.

3.2 The trial function which satisfies the initial and boundary condition for the state variable $\phi$ is chosen as,

$$\phi(x,t) = \phi(x,t)\Big|_{t=0} + \sum_{\ell=1}^{L} \sum_{m=1}^{M} a_{\ell m} \cos \frac{k_\ell \pi x}{2a} \; t^m$$

$$= \phi_o(x) + \sum_{\ell=1}^{L} \sum_{m=1}^{M} a_{\ell m} \cos \frac{k_\ell \pi x}{2a} \; t^m \qquad (3.5)$$

$a_{\ell m}$ are expansion coefficients which are constant.

In order to satisfy the boundary condition that the flux goes to zero at the outside surface for all t, $k_\ell$ have to be odd numbers. Thus,

$$k_\ell = 2\ell - 1 \; ; \quad \ell = 1,2,3,\ldots \; .$$

This trial function also satisfies the initial condition that,

$$\phi(x,0) = \phi_o(x) = A_o \cos \frac{\pi x}{2a} \; .$$

In the present problem M is chosen equal to 2 i.e. a second degree polynomial in time is taken for ease of computation. Choice of L was based on less than 1% variation which led to 12 term expansion. Similarly the trial function for the state variable N is constructed as,

$$N(x,t) = N(x,t)\Big|_{t=0} + \sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{k_\ell \pi x}{2a} \; t^m$$

$$= N_o + \sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{k_\ell \pi x}{2a} \; t^m \qquad (3.6)$$

where, $b_{\ell m}$ = constant expansion coefficients

and $k_\ell = 2\ell - 1 \; ; \quad \ell = 1,2,3,\ldots.. \; .$

This trial function satisfies the initial condition,

$$N(x,0) = N_o \ .$$

3.3 a) The performance index as given in equation 3.4 is very difficult to handle because of the nonlinear product term of the state variables N and $\phi$ and hence product of the expansion coefficients $a_{\ell m}$ and $b_{\ell m}$ will appear in the performance index.

From equation 3.2 it can be written

$$N\phi = -\frac{1}{\sigma_a}\frac{\partial N}{\partial t} = -k\frac{\partial N}{\partial t} \tag{3.7}$$

where, $k = \frac{1}{\sigma_a}$ .

Putting 3.7 in equation 3.4, the performance index will be dependent only on one state variable namely the number density N.

$$\text{P.I.} = \int_0^T \int_0^a \left[ \alpha_1 \left(k\frac{\partial N}{\partial t} + \frac{P}{a}\right)^2 + \alpha_2 \left(k\frac{\partial}{\partial x}\left(\frac{\partial N}{\partial t}\right)\right)^2 dx\, dt \right] \tag{3.8}$$

This performance index is easier to tackle since only one of the unknown coefficients of expansion namely $b_{\ell m}$ appear in it.

3.3 b) The constant power density constraint is

$$\int_0^a dx\, N\,\phi = P(t).$$

Once again eliminating the nonlinear product term $N\phi$ by using 3.7

$$-\int_0^a dx \left(\frac{1}{\sigma_a}\frac{\partial N}{\partial t}\right) = P(t) .$$

Using the polynomial expansion given in 1.5 for P(t) and the trial function for N as given in 3.6,

$$-\frac{1}{\sigma_a}\int_0^a \sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos\frac{k_\ell \pi x}{2a} . m\, t^{m-1}\, dx = \sum_{m=1}^{M} P_m t^{m-1} .$$

Now equating coefficients of like powers of t,

$$P_m = -\frac{m}{\sigma_a}\sum_{\ell=1}^{L} b_{\ell m}\int_0^a \cos\frac{k_\ell \pi x}{2a}\, dx \qquad (3.9)$$

$$m = 1,2,3,\ldots,M$$

After carrying out the integration in 3.9,

$$P_m = \frac{2am}{\pi\sigma_a}\sum_{\ell=1}^{L} b_{\ell m}\frac{(-1)^\ell}{k_\ell} \qquad m = 1,2,3,\ldots M$$

with little algebraic manipulation,

$$b_{1m} = H_m + \sum_{\ell=2}^{L} b_{\ell m}\frac{(-1)^\ell}{k_\ell} \qquad m = 1,2,3,\ldots M \qquad (3.10)$$

where

$$H_m = -\frac{P_m \sigma_a \pi}{2am} .$$

Thus the integral constraint is converted into a set of equivalent algebraic equations. The equation 3.10 tells that out of L × M variables, M variables ($b_{1m}$, m = 1,2,...,M) are constrained and the rest (L × M – M) are independent variables.

3.3 c) The trial function for N given in 3.6 after some algebraic manipulation becomes,

$$N = N_o + \sum_{\ell=2}^{L} \sum_{m=1}^{M} b_{\ell m} G_{\ell m} + \sum_{m=1}^{M} H_m \cos\frac{\pi x}{2a} t^m$$

where,

$$G_{\ell m} = \cos\frac{k_\ell \pi x}{2a} t^m + \frac{(-1)^\ell}{k_\ell} \cos\frac{\pi x}{2a} t^m$$

So,

$$\frac{\partial N}{\partial t} = \sum_{\ell=2}^{L} \sum_{m=1}^{M} b_{\ell m} G_{\ell mt} + \sum_{m=1}^{M} H_m \cos\frac{\pi x}{2a} \cdot mt^{m-1} \qquad (3.11)$$

and,

$$\frac{\partial}{\partial x}\left(\frac{\partial N}{\partial t}\right) = \sum_{\ell=2}^{L} \sum_{m=1}^{M} b_{\ell m} G_{\ell mtx} + \sum_{m=1}^{M} H_m\left(-\frac{\pi}{2a}\right)\sin\frac{\pi x}{2a}\, mt^{m-1}$$

$$\ldots \qquad (3.12)$$

$G_{\ell mt}$ represents differentiation of $G_{\ell m}$ with respect to t and $G_{\ell mtx}$ represents differentiation of $G_{\ell m}$ with respect to t and x both. They are given as

$$G_{\ell mt} = \cos\frac{k_\ell \pi x}{2a} mt^{m-1} + \frac{(-1)^\ell}{k_\ell} \cos\frac{\pi x}{2a} mt^{m-1}$$

$$G_{\ell mtx} = \left(-\frac{k_\ell \pi}{2a}\right) \sin\frac{k_\ell \pi x}{2a} mt^{m-1} + \frac{(-1)^\ell}{k_\ell}\left(-\frac{\pi}{2a}\right) \sin\frac{\pi x}{2a} mt^{m-1}$$

After substitution of 3.11 and 3.12 in the P.I. as given by

$$P.I. = \int_0^T \int_0^a \{ \alpha_1 k^2 \left( \sum_{\ell=2}^{L} \sum_{m=1}^{M} b_{\ell m} G_{\ell m t} + \sum_{m=1}^{M} H_m \cos\frac{\pi x}{2a} mt^{m-1} + \frac{P}{a} \right)^2$$

$$+ \alpha_2 k^2 \left( \sum_{\ell=2}^{L} \sum_{m=1}^{M} b_{\ell m} G_{\ell m t x} + \sum_{m=1}^{M} H_m \left(-\frac{\pi}{2a}\right) \sin\frac{\pi x}{2a} mt^{m-1} \right)^2 \} \, dx.dt \quad \ldots \quad (3.13)$$

To obtain the optimal solution, partial derivatives of the performance index as given by 3.13 w.r.t. all the independent variables ($b_{\ell m}$, $\ell = 2,3,\ldots,L$ ; $m = 1,2,\ldots,M$) are equated to zero.

$$\frac{\partial\, P.I.}{\partial b_{\ell m}} = 0 \; , \; \ell = 2,3,\ldots,L \; ; m = 1,2,\ldots,M \; . \qquad (3.14)$$

For the present problem the above conditions reduce to a set of $(L-1) \times M$ linear algebraic equations.

After some algebraic simplification these equations take the form,

$$k^2 \alpha_1 b_{\ell m} (G_{\ell m t}, G_{\ell m t}) + k^2 \alpha_1 \sum_{\substack{i \neq \ell \\ j \neq m}} \sum b_{ij} (G_{\ell m t}, G_{ijt})$$

$$+ \alpha_1 k(P, G_{nmt}) + \alpha_1 k^2 \sum_{m=1}^{M} (H_m \cos\frac{\pi x}{2a} mt^{m-1}, G_{\ell m t}) +$$

$$+ \alpha_2 k^2 (G_{\ell m t x}, G_{\ell m t x}) + \alpha_2 \sum_{\substack{i \neq \ell \\ j \neq m}} \sum b_{ij} (G_{\ell m t x}, G_{ijtx}) +$$

$$+ \alpha_2 \sum_{m=1}^{M} (H_m \left(-\frac{\pi}{2a}\right) \sin\frac{\pi x}{2a} \; mt^{m-1}, G_{\ell m t x}) = 0 \qquad (3.15)$$

$$\ell = 2,3,\ldots,L$$
$$m = 1,2,\ldots,M$$

where, the inner product $(x,y)$ denotes integration of these variables over the entire space and time. It is interesting to note that $(P, G_{\ell mt}) = 0$ as proved below,

$$\int_0^T \int_0^a P\, G_{\ell mt}\, dx\, dt = \int_0^T \int_0^a P \left[ \cos \frac{k_\ell \pi x}{2a} \,.\, m t^{m-1} + \frac{(-1)^\ell}{k} \cos \frac{\pi x}{2a} \,.\, m t^{m-1} \right] dx.\ dt$$

$$= P\, T^M \left( \frac{2a}{\pi k_\ell} (-1)^{\ell-1} + (-1)^\ell \frac{2a}{k_\ell \pi} \right) = 0$$

The algebraic equations finally take the form

$$\alpha_1 b_{\ell m}(G_{\ell mt}, G_{\ell mt}) + \alpha_1 \sum_{\substack{i \neq \ell \\ j \neq m}} \sum b_{ij} (G_{\ell mt}, G_{ijt})$$

$$+ \alpha_1 \sum_{m=1}^{M} (H_m \cos \frac{\pi x}{2a} m t^{m-1} , G_{\ell mt}) + \alpha_2 (G_{\ell mtx}, G_{\ell mtx})$$

$$+ \alpha_2 \sum_{\substack{i \neq \ell \\ j \neq m}} \sum b_{ij}(G_{\ell mtx}, G_{ijtx}) + \alpha_2 \sum_{m=1}^{M} (H_m(-\frac{\pi}{2a}) \sin \frac{\pi x}{2a} m t^{m-1} ,$$

$$G_{\ell mtx}) = 0 \quad (3.16)$$

$$\ell = 2,3,\ldots, L$$

$$m = 1,2,\ldots,M$$

This set of linear algebraic equations are solved to obtain the independent variables $b_{\ell m}$. ($\ell = 2,3,\ldots, L$ ; $m = 1,2,\ldots,M$). The unconstrained variables $b_{1m}$ ($m = 1,2,k\ldots, M$) may be calculated from the constraint relation 3.10.

When all $b_{\ell m}$ are known, the closed form solution for $N(x,t)$ is given by equation 3.6. The neutron flux $\phi(x,t)$ is calculated from the equation

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N$$

or,

$$\phi = - \left(\frac{\partial N}{\partial t}\right) / \sigma_a N \tag{3.17}$$

The whole calculation is carried out for a certain reactor operating time $T_1$ days. The poison cross-section is found out when N and $\phi$ are known from the equation 3.1 as

$$\Sigma_{ap} = D \frac{\partial^2 \phi}{\partial x^2} / \phi + (\nu \sigma_f - \sigma_a) N .$$

If the value of $\Sigma_{ap}$ is greater than a preset designed value, the operating time is increased to $T_2$ days by an amount $\Delta t$. The procedure is repeated until the poison cross-section reduces to the preset design value. Since the end value of the poison cross-section was not available for the present problem, the reactor was operated upto a maximum of T = 325 days and the final burnup data was obtained at T = 325 days.

3.4 Results of the depletion problem in a slab Nuclear Reactor of width 200 cm fueled with $U_{235}$ $\left(\frac{N_{25}}{N_{28}} = .4\right)$ with initial multiplication factor of 1.032 obtained in this chapter are discussed in this section. Input parameters are same as in Becker's problem[15] only P is considered

as a first degree polynomial in time as opposed to a constant in Becker's problem.

$$D = 1 \text{ cm} \quad (\text{constant})$$

$$\sigma_a = 200 \text{ barns } (1 \text{ barn} = 10^{-24} \text{ cm}^2)$$

$$\nu \sigma_f = 400 \text{ barns}$$

$$\phi_o = A_o \cos \frac{\pi x}{2a}$$

where $A_o = 6.284 \times 10^{13} \text{ cm}^{-2} \text{ sec}^{-1}$

$$N_o = 2 \times 10^{19} \text{ cm}^{-3}$$

$$\Sigma_{ap} = 3.7605 \times 10^{-3} \text{ cm}^{-1}$$

$$T_1 = 200 \text{ days}$$

$$\Delta t = 25 \text{ days}$$

$$P(t) = P_1 + P_2 t$$

$$P_1 = 6.9 \times 10^{39} \text{ cm}^{-4} \text{ sec}^{-1}$$

$$P_2 = P_1/(10t) .$$

The summation indices are N = 12 and M = 2.

The weight factors are $\alpha_1 = 1$ and $\alpha_2 = 50$ .

The choice of the polynomial for power leads to a 10% increase in power but this is done in order to facilitate the solution and to see how the nuclear parameters change with a polynomial power constraint. The table 3.1 shows the neutron flux distribution and

fuel distribution over the space for different operating times and these are plotted in figure 3.1. The figure 3.2 shows the variation of poison cross-section with reactor operating time T. The poison cross-section here also goes down steadily due to the assumption that no $Pu^{239}$ is produced during the operation of the reactor. The variation in N and $\phi$ is similar to the results of the previous chapter but magnitudes cannot be compared as the present case deals with spatial power smoothening throughout core life time as opposed to constant flux in small time step intervals of the previous chapter. There are no experimental results available for simple slab geometry investigated in this thesis . The input data was not varied to consider practical situations, due to lack of computer availability.

| x cm | $\phi \times 10^{-13}$ T=200 days | $\phi \times 10^{-13}$ T=250 days | $\phi \times 10^{-13}$ T=300 days | $N \times 10^{-19}$ T=200 days | $N \times 10^{-19}$ T=250 days | $N \times 10^{-19}$ T = 300 days |
|---|---|---|---|---|---|---|
| 0 | 7.011 | 7.680 | 8.462 | 1.607 | 1.503 | 1.397 |
| 10 | 6.993 | 7.657 | 8.432 | 1.609 | 1.505 | 1.399 |
| 20 | 6.781 | 7.406 | 8.133 | 1.619 | 1.519 | 1.416 |
| 30 | 6.480 | 7.051 | 7.710 | 1.636 | 1.539 | 1.441 |
| 40 | 6.116 | 6.624 | 7.203 | 1.657 | 1.566 | 1.473 |
| 50 | 5.580 | 6.004 | 6.481 | 1.685 | 1.604 | 1.519 |
| 60 | 5.096 | 5.447 | 5.837 | 1.717 | 1.642 | 1.565 |
| 70 | 4.477 | 4.747 | 5.044 | 1.754 | 1.689 | 1.622 |
| 80 | 3.811 | 4.006 | 4.217 | 1.796 | 1.742 | 1.687 |
| 90 | 3.064 | 3.191 | 3.326 | 1.842 | 1.801 | 1.758 |
| 100 | 0.000 | 0.000 | 0.000 | 1.999 | 1.999 | 1.999 |

TABLE 3.1

FUEL AND FLUX DISTRIBUTION

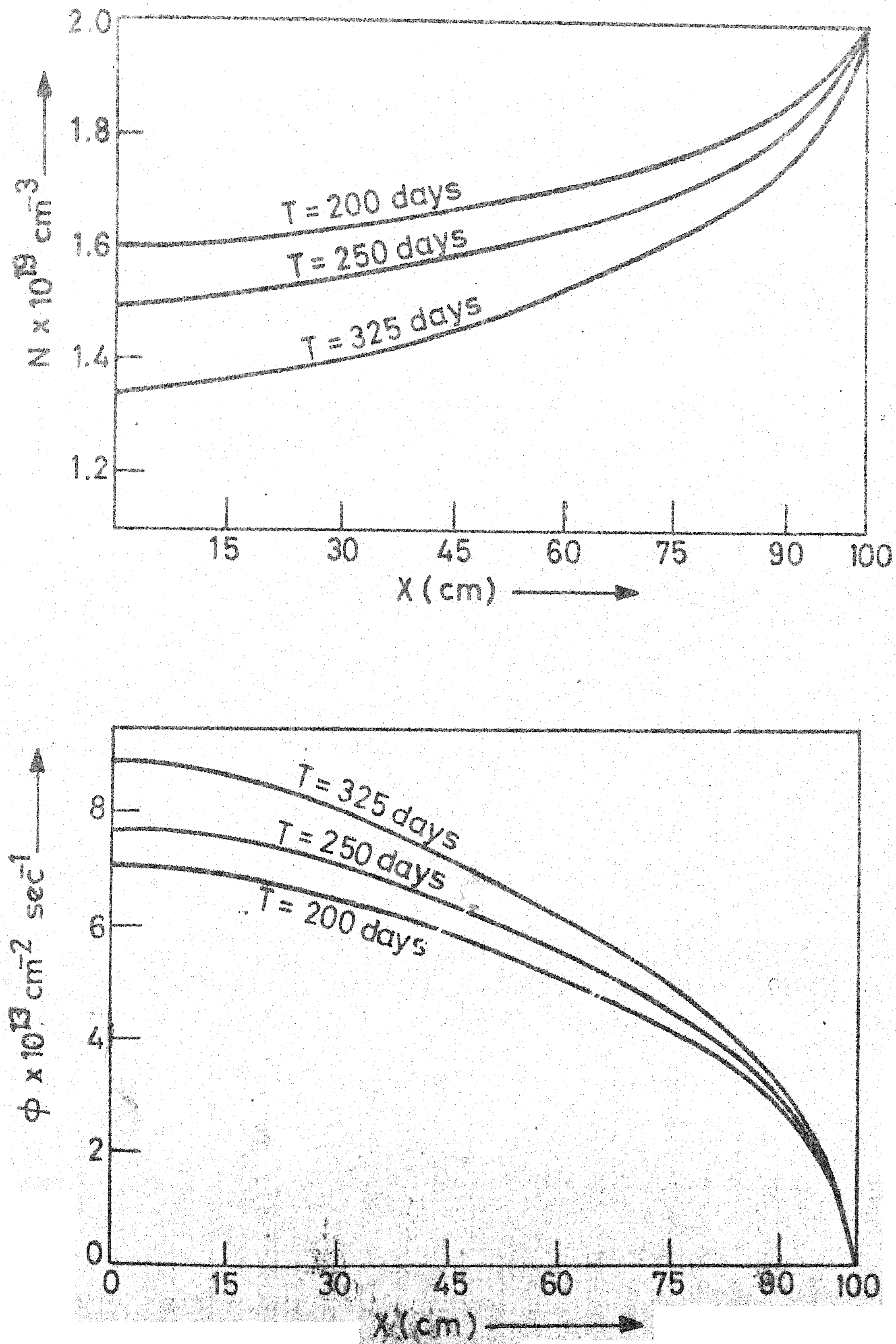

2.0
1.8
1.6
1.4
1.2
N x 10^19 cm^-3
T = 200 days
T = 250 days
T = 325 days
15
30
45
60
75
90
100
X (cm)
8
6
4
2
0
φ x 10^13 cm^-2 sec^-1
T = 325 days
T = 250 days
T = 200 days
0
15
30
45
60
75
90
100
X (cm)

FIG. 3.1

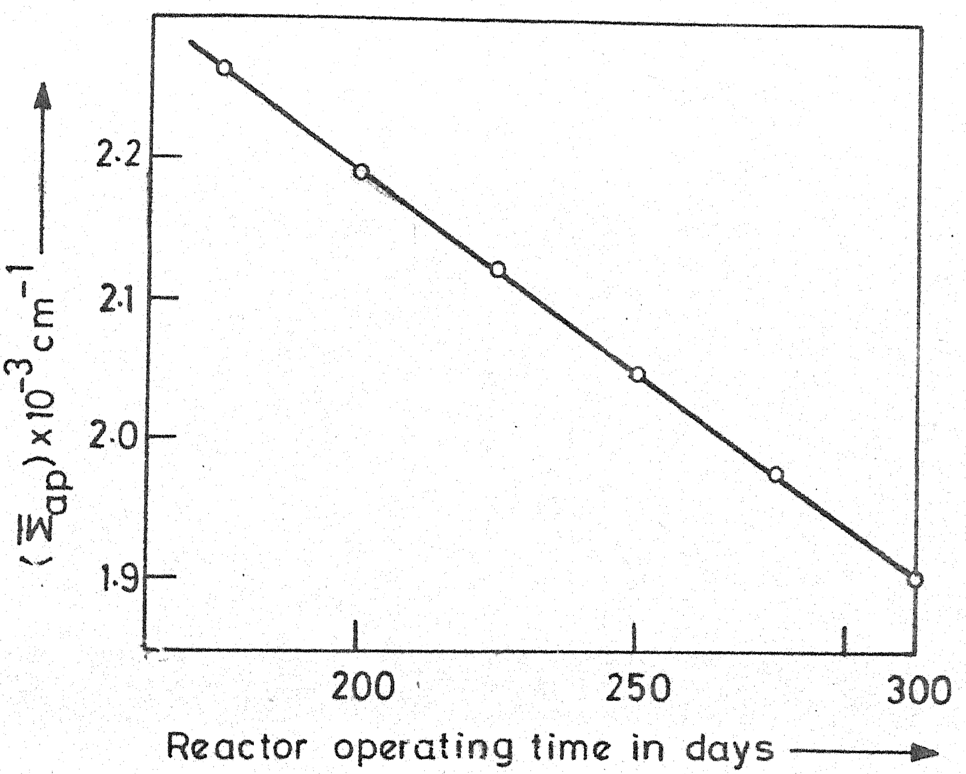

2.2
2.1
2.0
1.9
$(\overline{\Sigma}_{ap}) \times 10^{-3} \text{cm}^{-1}$
200
250
300
Reactor operating time in days

FIG. 3-2

# CHAPTER - 4

4.1 Fuel management is essentially the study of fuel loading schedules. Some of these schemes may require changing the location of the fuel elements within the core according to a predetermined schedule. Although these schemes may increase the burnup and reduce the expenditure of fuel, they may add to the cost of reactor operation, especially if shutdown is necessary for rearranging the fuel. So, careful economic analysis has to be made to determine the merit of the various scheduling schemes proposed by optimization studies. In the present chapter general fuel loading schemes are described and batch loading scheme is investigated in studying the fuel depletion problem.

Batch Loading :

The reactor is loaded uniformly with complete core at one time and this is irradiated without moving the fuel elements. When the system ceases to be critical the whole core is discharged and reloaded. As a modification of the simple batch irradiation scheme, the core may be divided into two or more radial zones, with fuel of different enrichment in each zone.

Centre to Outside Loading :

From time to time, fresh fuel material is added to the centre of the core, where the leakage of neutrons is low and hence

their importance is high and progressively moved to outer radial positions. The spent fuel is discharged from the outer radius of the core.

Outside to Central Loading :

In this procedure, the fresh fuel is charged near the outer edge and progressively moved towards the centre from where it is discharged.

Bidirectional Loading :

In bidirectional loading, charging and discharging of the fuel are achieved by pushing relatively short fuel bundles through the core from one side of the reactor to the other, but from opposite directions in adjacent rows of fuel channels.

4.2 The reactor is operated according to the desired criteria of spatial power smoothening as in chapter 3 in the present study. The same depletion problem is considered here also. The operation of the reactor is stopped after time T which varied from 200 days to 300 days in steps of 50 days for refuelling. The comparatively more depleted fuel zone is replaced by fresh fuel of the uniform atom density corresponding to the concentration after T days of irradiation at the spatial location of x = 60 cm as shown by the point B in figure 4.1.

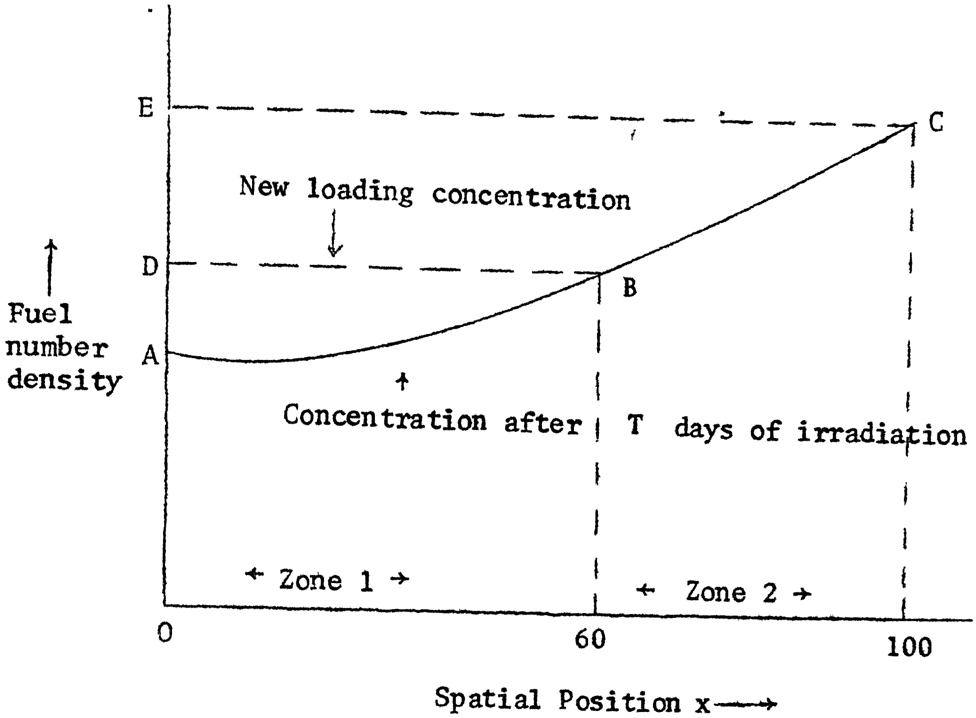



Figure 4.1.

The comparatively less depleted fuel zone (zone 2) is not changed. For further reactor operation the initial fuel configuration is now constant value in zone 1 as indicated by the DB line and the depleted values as given by the curve BC in zone 2. Mathematically this can be represented as,

$$\bar{N}(x)\big|_{t=0} = N(60)\big|_{t=200} \times \{U_0(x) - U_{60}(x)\}$$

$$\text{(after refuelling)} + N(x)\big|_{t=200} \times U_{60}(x) \qquad 4.1$$

$\bar{N}$ = New fissile-number density.

where the Unit step function U is defined as

$$U_0(x) = 0 \qquad x < 0$$

$$U_0(x) = 1 \qquad x > 0$$

$$U_{60}(x) = 0 \qquad x < 60$$

$$U_{60}(x) = 1 \qquad x > 60$$

4.3 The new trial function for fuel density is given by

$$\bar{N}(x,t) = \bar{N}(x,0) + \sum_{\ell=1}^{L} \sum_{m=1}^{L} \bar{b}_{\ell m} \cos\frac{K_\ell \pi x}{2a}\, t^m \qquad 4.2$$

$$K_\ell = 2\ell - 1; \quad \ell = 1,2,3,\ldots$$

where $\bar{b}_{\ell m}$ are the expansion coefficients.

Following the procedure outlined in chapter 3 the equivalent algebraic form of the integral constraint on power is given by

$$\bar{b}_{1m} = H_m + \sum_{\ell=2}^{L} \bar{b}_{\ell m} \frac{(-1)^\ell}{K_\ell} \qquad 4.3$$

$$m = 1,2,\ldots,M$$

where $H_m = \dfrac{-P_m \sigma_a \pi}{2am}$.

Incorporating 4.3 in 4.2 and after some algebraic manipulation,

$$\bar{N}(x,t) = \bar{N}(x,0) + \sum_{\ell=2}^{L} \sum_{m=1}^{M} \bar{b}_{\ell m} G_{\ell m}(x,t) + \sum_{n=1}^{M} H_n \cos\frac{\pi x}{2a}\, t^n$$

where,

$$G_{\ell m} = \cos\frac{K_\ell \pi x}{2a}\, t^m + \frac{(-1)^\ell}{\ell} \cos\frac{\pi x}{2a}\, t^m \qquad 4.4$$

The spatial power smoothening being the desired criteria, the performance index remains the same as in chapter 3. Thus

$$P.I. = \int_0^T \int_0^a [\alpha_1 (K \frac{\partial \bar{N}}{\partial t} + \frac{P}{a})^2 + \alpha_2 (K \frac{\partial}{\partial x} \frac{\partial \bar{N}}{\partial t})^2 ] dx\, dt \qquad 4.5$$

The function $\bar{N}(x,0)$ is continuous as seen in the figure 4.1, but its derivative $\frac{\partial \bar{N}}{\partial x}$ is not continuous. To avoid the point of discontinuity integrations are performed upto a small neighborhood $\varepsilon$ of the point $x = 60$ cm. The performance index may then be written as

$$P.I. = \int_0^T \int_0^{60-\varepsilon} \{\alpha_1 (K \frac{\partial N}{\partial t} + \frac{P}{a})^2 + \alpha_2 (K \frac{\partial}{\partial x} (\frac{\partial N}{\partial t}))^2\}^2 dx\, dt$$

$$+ \int_0^T \int_{60+\varepsilon}^{100} \{\alpha_1 (K \frac{\partial N}{\partial t} + \frac{P}{a})^2 + \alpha_2 (K \frac{\partial}{\partial x} (\frac{\partial N}{\partial t}))^2\} dx\, dt \qquad 4.6$$

where $\varepsilon$ is a very small quantity and is taken to be 0.01 for the computations performed in this chapter. The performance index in this particular form does not produce any computational difficulty.

The expansion coefficients $\bar{b}_{\ell m}$ are solved by the same calculus of variation technique as in chapter 3. When $\bar{b}_{\ell m}$ are solved, closed-form solution for $\bar{N}$ is achieved through equation 4.2. The calculation of neutron flux and poison cross-section is exactly the same as in chapter 3.

4.4 The reactor description and the input parameters are same as in chapter 3.

The table 4.1 shows the neutron flux and fuel distribution over space for different operating times T after refuelling.

They are plotted in figure 4.2. The figure 4.3a shows the variation of poison cross-section with operating time T. As observed in earlier chapters, the poison cross-section steadily goes down with time. This is because of the fact that it is assumed that there is no plutonium production from $U_{238}$.

The objective for refuelling may be two fold i) to increase the burnup of the fuel ii) to increase the overall corelife of the reactor But to ascertain whether a particular fuel schedule should be adopted, careful economic analysis should be carried out first. In the present problem, the comparatively more depleted fuel-zone is replaced by a uniform fuel density of higher concentration. The less depleted fuel zone near the core end is not replaced. So, this allows effective burning burning of the fuel near the end which is not realised if the whole core is refuelled. Due to the introduction of fresh fuel, the amount of poison cross-section for controlling criticality increases and hence the reactor may be operated in the critical condition for longer time.

An approximate estimate of the fuel cost of the newly introduced fuel may be obtained in terms of reactor operation days. From fig. 4.1 it is given as

$$\text{Fuel Cost} = \frac{\text{Area enclosed by ABD}}{\text{Area enclosed by AEC}} \times \text{Time of refuelling} \qquad 4.7$$

For the present study, refuelling is done after 200, 250, 300 days and the fuel costs of the newly introduced fuel in terms of days are

25, 31, 37.5 days respectively. Now if the reactor is not refuelled the poison cross-section reduces to $1.901 \times 10^{-3}$ $cm^{-1}$ after 300 days of operation. As seen from figure 4.3, the reactor operates for (200 + 133) = 333 days in total when refuelling is done at T = 200 days before it reaches the poison cross-section $1.901 \times 10^{-3}$ $cm^{-1}$. Similarly, the reactor operates for (250 + 88) = 338 days if refuelled at T = 250 days and (300 + 45) = 345 days if refuelled at T = 300 days before the poison cross-section goes down to $1.901 \times 10^{-3}$ $cm^{-1}$. Therefore, the reactor operates for 33, 38 and 45 days more if refuelled at T = 200, 250, and 300 days respectively. The gain due to refuelling may be obtained by subtracting this extra operation days from the fuel cost of the new fuel introduced at the time of the refuelling.

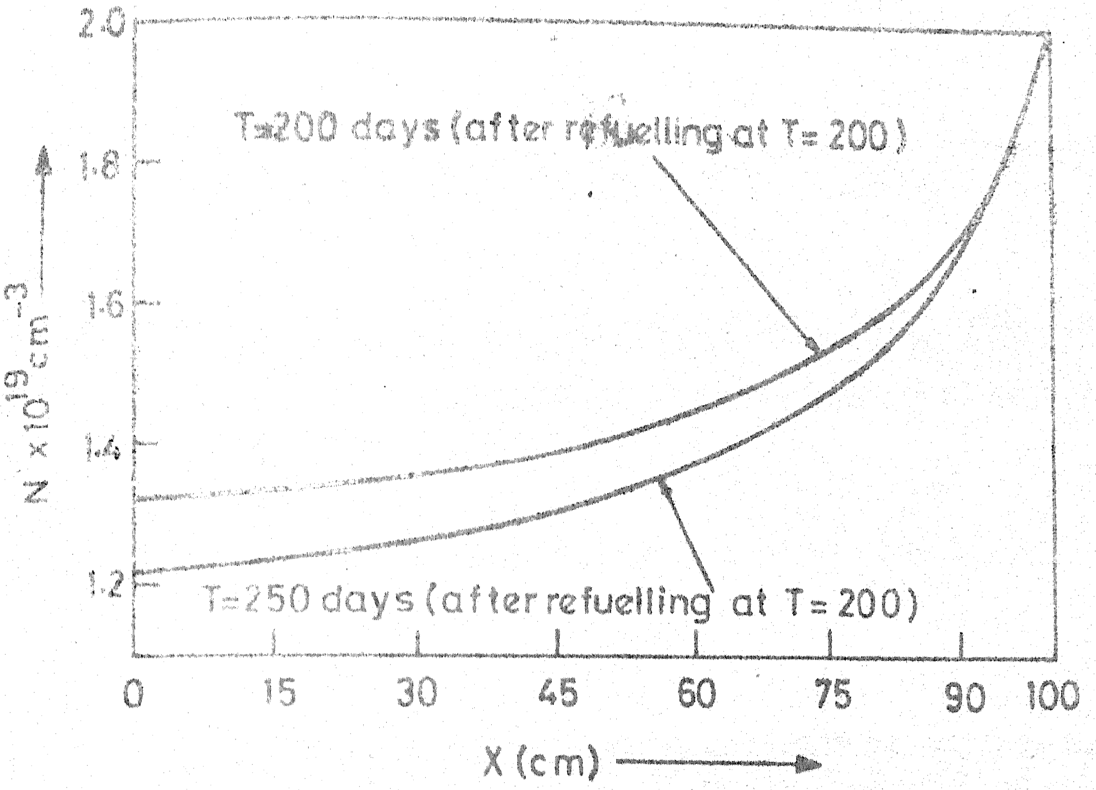
Thus the gain achieved due to the partial refuelling at T = 200 days is given as (33 - 25) = 8 days. Similarly the different gains for refuelling at T = 250, 300 days are 7 and 7.5 days. The number of days gained vs. refuelling time may be plotted from where the optimum time of refuelling can be chosen for maximum gain. In the present study the gain appears to be almost constant with time of refuelling according to Fig. 4.3. This is due to the rough estimate of fuel cost calculated on area basis and also due to the fact no plutonium production is considered. So in the present case it is very difficult to make a decision on refuelling time for achieving a maximum gain of reactor operating time.

| x cm | $\phi \times 10^{-13}$ T = 200 | $\phi \times 10^{-13}$ T = 250 | $N \times 10^{-19}$ T = 200 | $N \times 10^{-19}$ T = 250 |
|---|---|---|---|---|
| 0 | 8.510 | 9.463 | 1.324 | 1.219 |
| 10 | 8.486 | 9.431 | 1.325 | 1.222 |
| 20 | 8.217 | 9.103 | 1.336 | 1.236 |
| 30 | 7.836 | 8.640 | 1.353 | 1.256 |
| 40 | 7.377 | 8.085 | 1.374 | 1.283 |
| 50 | 6.706 | 7.291 | 1.403 | 1.321 |
| 60 | 6.060 | 6.541 | 1.439 | 1.364 |
| 70 | 5.205 | 5.554 | 1.509 | 1.444 |
| 80 | 4.298 | 4.537 | 1.592 | 1.538 |
| 90 | 3.350 | 3.496 | 1.685 | 1.643 |
| 100 | 0.000 | 0.000 | 1.999 | 1.999 |

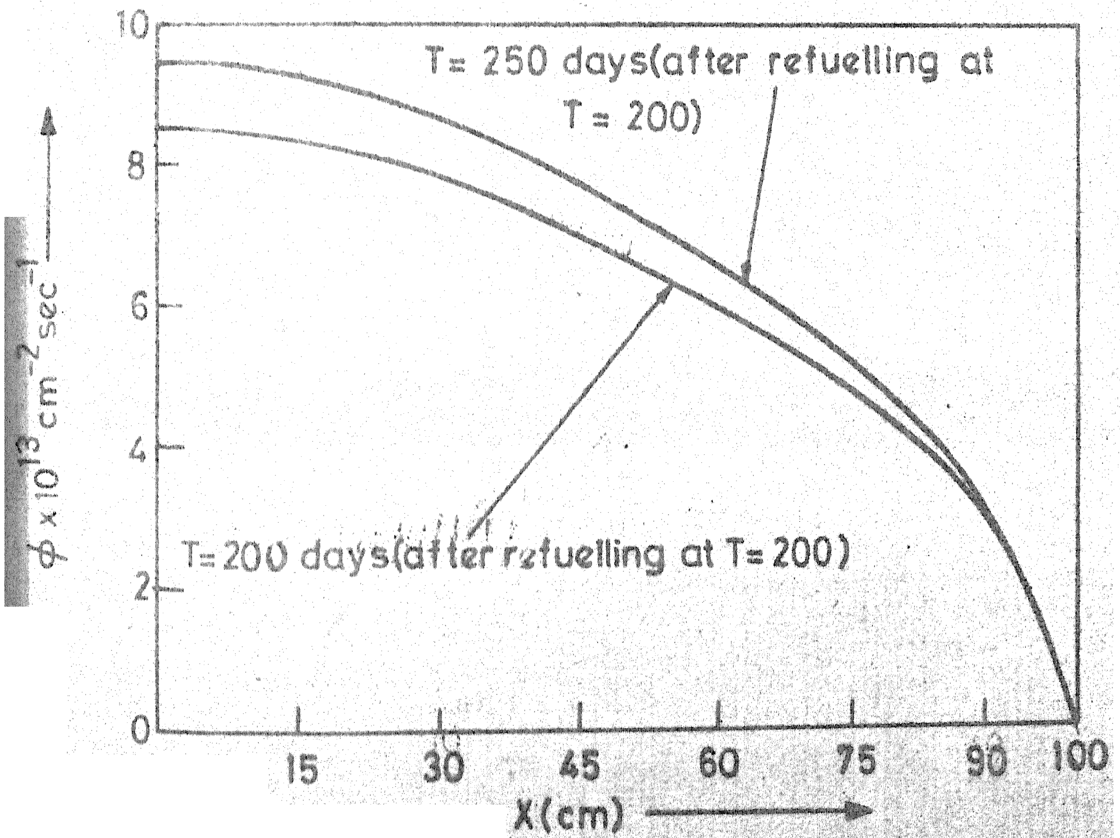
TABLE 4.1

FUEL AND FLUX DISTRIBUTION

(AFTER REFUELLING)

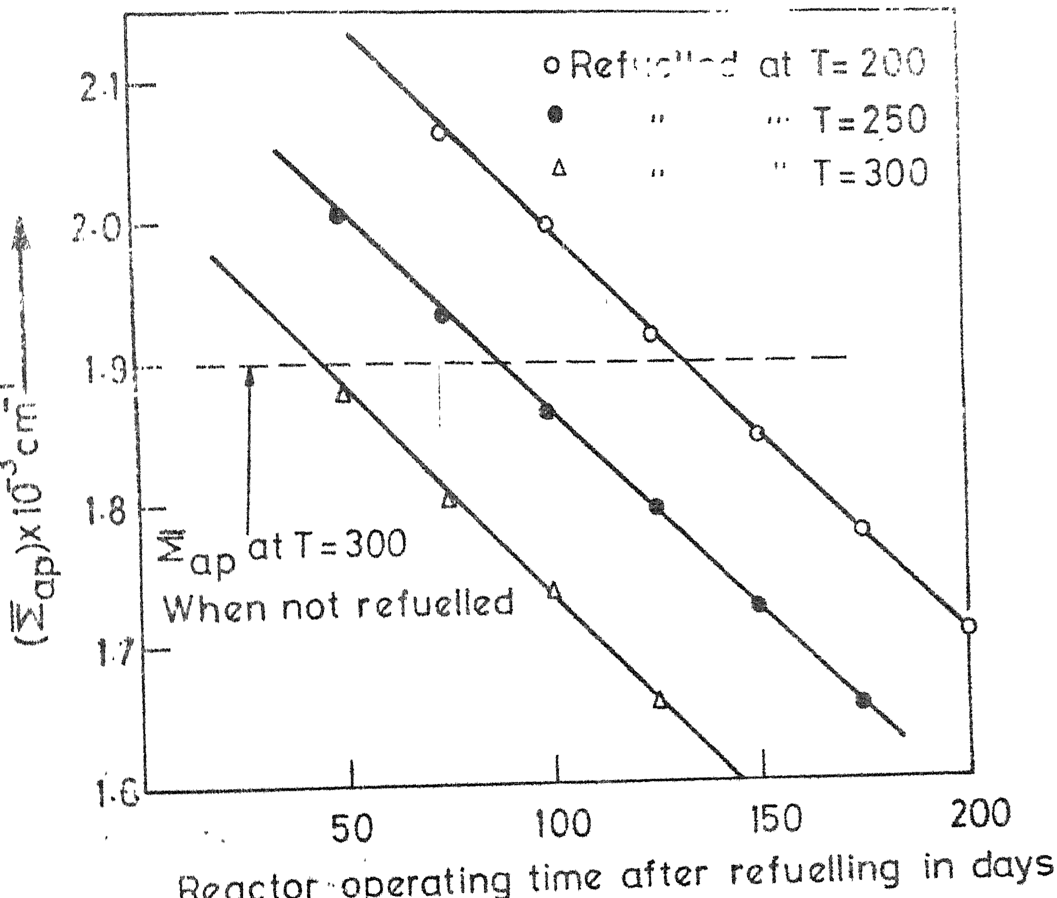


T=200 days (after refuelling at T= 200)
T=250 days (after refuelling at T= 200)
$N \times 10^{19} cm^{-3}$
X (cm)

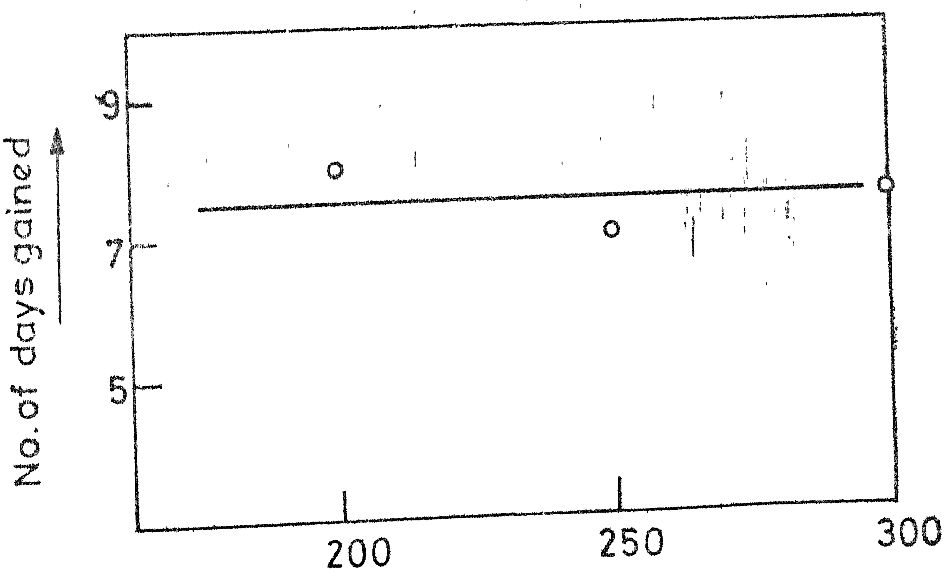
T= 250 days(after refuelling at T = 200)
T=200 days(after refuelling at T= 200)
$\phi \times 10^{13} cm^{-2} sec^{-1}$
X(cm)


FIG. 4.2



o Refuelled at T = 200
" " T = 250
" " T = 300
2.1
2.0
1.9
1.8
1.7
1.6
$(\overline{\Sigma}_{ap}) \times 10^{-3} cm^{-1}$
$\overline{\Sigma}_{ap}$ at T = 300
When not refuelled
50
100
150
200
Reactor operating time after refuelling in days


(a)



9
7
5
No. of days gained
200
250
300
Refueling time in days


(b)

FIG. 4.3

# CHAPTER - 5

5.1 This chapter makes an approach to the solution of the depletion problem as described mathematically by the following equations with a desired criteria of achieving minimal control effort. The minimal control effort is accomplished by variational technique as outlined in chapter 3. The governing equations are,

$$D \frac{\partial^2 \phi}{\partial x^2} + (\nu \sigma_f - \sigma_a) N\phi - \Sigma_{ap} \phi = 0 \tag{5.1}$$

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N \tag{5.2}$$

$$\int_0^a dx \, N\phi = P(t) \tag{5.3}$$

The boundary conditions on N and $\phi$ are the same as in previous chapters. The performance index is chosen so as to minimize the parasitic capture of neutrons in the control material which will indirectly result in less fuel consumption. Mathematicaly this is given by,

$$P.I. = \int_0^t \int_0^a \Sigma_{ap} \, dx \, dt \tag{5.4}$$

where the poison cross-section $\Sigma_{ap}$ is the control variable. The fissile number density N and flux $\phi$ are the state variables. The optimal control problem is to find the values of N and $\phi$ for which the desired criteria given by equation 5.4 is achieved. Mathematically, the problem in this chapter is more difficult than the problem posed in chapter 3 because the control variable appears explicitly in the performance index and hence nonlinear equation is to be solved without further simplifications. The diffusion equation 5.1 may be used to find the desired values of fissile number density N and $\phi$ by substituting these variables for $\Sigma_{ap}$ in equation 5.4. Thus the performance index changes to,

$$P.I. = \int_0^T \int_0^a \left\{ \frac{D \frac{\partial^2 \phi}{\partial x^2}}{\phi} + (\nu \sigma_f - \sigma_a) N \right\} dx\, dt \qquad (5.5)$$

The nonlinearity arises because of the state variable $\phi$ occuring in the denominator and this causes complexity in the present problem. Note that the performance index in chapter 3 contained only the state variable N. But here it contains both the state variables N and $\phi$ as given by 5.5.

5.2 With the same basic philosophy as outlined in chapter 3, trial functions for the state variables $\phi$ and N are constructed in this chapter. They are given by equations 5.6 and 5.7. The constant power constraint as given by equation 5.3 puts restrictions

similar to equation 3.10 on the coefficients of expansion $b_\ell$ . Following the steps as shown in chapter 3 this condition given by equation 5.8 is obtained. Here time variation of flux is chosen as linear as opposed to the quadratic polynomial chosen in chapter 3 for mathematical simplicity . So, the governing equations are ,

Trial functions for $\phi$ and N:

$$\phi(x,t) = \phi_o(x) + \sum_{\ell=1}^{L} a_\ell \cos \frac{K_\ell \pi x}{2a} t \qquad (5.6)$$

$$N(x,t) = N_o + \sum_{\ell=1}^{L} b_\ell \cos \frac{K_\ell \pi x}{2a} t \qquad (5.7)$$

Equivalent algebraic equation for power constraint equation:

$$b_1 = -\frac{P_1 \pi\sigma_a}{2a} + \sum_{\ell=2}^{L} b_\ell \frac{(-1)^\ell}{K_\ell} \qquad (5.8)$$

Since both $\phi$ and N are involved in the performance index 5.5, it is essential to find a relation between the coefficients of expansion $a_\ell$ and $b_\ell$ in order to obtain complete solution.

5.3 The Galerkin method of weighted residuals is used to find out the relation between $a_\ell$ and $b_\ell$ . If the depletion equation 5.2 is used without linearization in the Galerkin method, the resulting relation between $a_\ell$ and $b_\ell$ will be nonlinear due to the product term of the state variables N and $\phi$ occuring in 5.2. To avoid this problem and to use the Galerkin method effectively linearization

of the depletion equation is done by assuming perturbation around initial values of the two state variables namely $N_o$ and $\phi_o$. Thus in the first order approximation the depletion equation may be written as

$$\frac{\partial}{\partial t}(N_o+\delta N) + \sigma_a(\phi_o + \partial\phi)\ (N_o + \delta N) = 0$$

where, $\delta N = N - N_o$ (5.9)

$\delta\phi = \phi - \phi_o$ (5.10)

Neglecting the product of $\delta\phi$ and $\delta N$,

$$\frac{\partial N_o}{\partial t} + \frac{\partial \delta N}{\partial t} + \sigma_a\ \phi_o\ \delta N + \sigma_a\ N_o\ \delta\phi + \sigma_a\ \phi_o\ N_o = 0$$

Using 5.9 and 5.10 for $\delta N$ and $\delta\phi$, the above equation reduces to

$$\frac{\partial N}{\partial t} + \sigma_a\ N_o\phi + \sigma_a\ \phi_o\ N - \sigma_a\ \phi_o\ N_o = 0 \tag{5.11}$$

The Galerkin method then yields the following equations,

$$\left(\frac{\partial N}{\partial t} + \sigma_a\ N_o\phi + \sigma_a\ \phi_o\ N - \sigma_a\phi_o\ N_o,\ \cos\frac{K_p\ \pi x}{2a}\right) = 0$$

$$P = 1,2,\ldots,L$$

where $\cos\frac{K_p\ \pi x}{2a}$ are the basis vectors of the trial functions. The inner product $(x,y)$ indicates integration over entire space and time. These inner products are easily calculated by employing orthogonality property of the cosine function. Except the first one they are not shown here.

$$(\sigma_a N_o \phi, \cos \frac{K_p \pi x}{2a}) = \int_0^T \int_0^a \{\sigma_a N_o [A_o \cos \frac{\pi x}{2a} + \sum_{\ell=1}^{L} a_\ell \cos \frac{K_\ell \pi x}{2a} t] \times$$

$$\times \cos \frac{K_p \pi x}{2a}\} \, dx \, dt$$

$$= \frac{a}{2} \sigma_a N_o A_o T \delta_{p1} + a_{p1} \frac{T^2}{4} a$$

where

$\delta_{p1} \equiv$ Kronecker delta

i.e. $\delta_{p1} = 1$ when $p = 1$

$= 0$ otherwise

After carrying out the integrations and simplifying the algebraic steps, the final relation arrived at is,

$$a_\ell \frac{T}{2} = - b_\ell (1 + \frac{T}{2}) - 3\sigma_a N_o A_o \delta_{\ell 1} \quad (\ell=1,2,\ldots L) \qquad (5.12)$$

5.4 Using 5.12 and 5.8 in 5.6 and after some algebraic manipulation the expression for flux becomes,

$$\phi = \phi_o + \sum_{\ell=2}^{L} -b_\ell G_\ell + Y_1 \cos \frac{\pi x}{2a} t = \phi_o + \Delta\phi \qquad (5.13)$$

where $$\Delta\phi = \sum_{\ell=2}^{L} - b_\ell G_\ell + Y_1 \cos \frac{\pi x}{2a} t \qquad (5.14)$$

The quantities $G_\ell$ and $Y_1$ are defined as,

$$G_\ell = (1 + \frac{T}{2}) \frac{2}{T} [\cos \frac{K_\ell \pi x}{2a} + \frac{(-1)^\ell}{K_\ell} \cos \frac{\pi x}{2a}] t$$

and

$$Y_1 = \left[\left(1 + \frac{T}{2}\right) \frac{P_1 \pi \sigma_a}{2aT} - \frac{6\sigma_a N_o A_o}{T}\right]$$

So, $\phi_{xx} = \phi_{oxx} + \Delta\phi_{xx}$

where $$\Delta\phi_{xx} = \sum_{\ell=2}^{L} - b_\ell \, G_{\ell xx} - Y_1 \, t \left(\frac{\pi}{2a}\right)^2 \cos\frac{\pi x}{2a} \qquad (5.15)$$

where suffix xx denotes double differentiation with respect to x. $G_{\ell xx}$ is given as

$$G_{\ell xx} = \left(1 + \frac{T}{2}\right) \frac{2}{T} \left[ -\left(\frac{K_\ell \pi}{2a}\right)^2 \cos\frac{K_\ell \pi x}{2a} - \frac{(-1)^\ell}{K_\ell} \left(\frac{\pi}{2a}\right)^2 \cos\frac{\pi x}{2a}\right] t$$

Hence,

$$\frac{\phi_{xx}}{\phi} = \frac{\phi_{oxx} + \Delta\phi_{xx}}{\phi_o + \Delta\phi}$$

with the assumption $\Delta\phi$ is small compared to $\phi_o (\Delta\phi << \phi_o)$, the expression in the right hand side can be approximated to,

$$\frac{\phi_{xx}}{\phi} = \frac{\{\phi_{oxx} + \Delta\phi_{xx}\} \{1 - \frac{\Delta\phi}{\phi_o}\}}{\phi_o}$$

Noting that

$$\phi_o = A_o \cos\frac{\pi x}{2a}$$

$$\phi_{oxx} = - A_o \left(\frac{\pi}{2a}\right)^2 \cos\frac{\pi x}{2a}$$

and also using 5.14 and 5.15 with some algebraic rearrangements,

$$\frac{\phi_{xx}}{\phi} = \{Y_2 + \sum_{\ell=2}^{L} -\frac{b_\ell}{A_o} G_{\ell xx} \sec \frac{\pi x}{2a}\} \{Y_3 + \sum_{\ell=2}^{L} \frac{b_\ell}{A_o} G_\ell \sec \frac{\pi x}{2a}\} \tag{5.16}$$

where,

$$Y_2 = -(\frac{\pi}{2a})^2 - Y_1 t (\frac{\pi}{2a})^2/A_o$$

$$Y_3 = 1 - Y_1 t/A_o$$

Now from equations 5.7 and 5.8 after some algebraic simplification,

$$N = N_o + \sum_{\ell=2}^{L} b_\ell \bar{G}_\ell + H_1 \cos \frac{\pi x}{2a} t \tag{5.17}$$

where

$$H_1 = \frac{-P_1 \pi \sigma_a}{2a}$$

$$\bar{G}_\ell = [\cos \frac{K_\ell \pi x}{2a} + \frac{(-1)^\ell}{K_\ell} \cos \frac{\pi x}{2a}] t$$

From 5.17 ,

$$\frac{\partial N}{\partial t} = \sum_{\ell=2}^{L} b_\ell \bar{G}_{\ell t} + H_1 \cos \frac{\pi x}{2a} \tag{5.18}$$

where suffix t denotes differentiation w.r.t. t.

Finally substituting 5.16 and 5.18 in the equation 5.5, the performance index now becomes,

$$\text{P.I.} = \int_0^t \int_0^a [\{y_2 + \sum_{\ell=2}^{L} -\frac{b_\ell}{A_o} G_{\ell xx} \sec \frac{\pi x}{2a}\} \{y_3 + \sum_{\ell=2}^{L} \frac{b_\ell}{A_o} G_\ell \sec \frac{\pi x}{2a}\}$$

$$+ (\nu\sigma_f - \sigma_a) \{\sum_{\ell=2}^{L} b_\ell \bar{G}_{\ell t} + H_1 \cos \frac{\pi x}{2a}\}] \, dx \, dt \qquad \ldots (5.19)$$

The solution of the minimal control problem is then given by,

$$\frac{\partial P.I.}{\partial b_\ell} = 0 \quad (\ell = 2,3,\ldots L) \tag{5.20}$$

This yields a set of linear algebraic equations since a linearized functional is used. These algebraic equations can be written as,

$$- \sum_{\ell=2}^{L} \frac{b_\ell}{A_o^2} \int_0^t \int_0^a G_\ell \, G_{ixx} \sec^2 \frac{\pi x}{2a} \, dx \, dt$$

$$- \frac{1}{A_o} \int_0^T \int_0^a y_3 \, G_{ixx} \sec \frac{\pi x}{2a} \, dx \, dt + \frac{1}{A_o} \int_0^T \int_0^a y_2 G_i \sec \frac{\pi x}{2a} \, dx \, dt$$

$$- \sum_{\ell=2}^{L} \frac{b_\ell}{A_o^2} \int_0^T \int_0^a G_{\ell xx} \, G_i \sec^2 \frac{\pi x}{2a} \, dx \, dt = 0 \tag{5.21}$$

$$i = 2,3,\ldots L$$

The different integrals involved in 5.21 are carried-out numerically. The solution of these (L-1) algebraic equations will give the coefficients of expansion $b_\ell$ ($\ell = 2,3,\ldots L$). When $b_\ell (\ell=2,3,\ldots L)$ are obtained $b_1$ is determined from the constraint equation 5.8. Since all the expansion coefficients $b_\ell$ are known, the closed form wolution for the state variable N is given by 5.7. The solution for flux is obtained from 5.2 as,

$$\phi = - \left(\frac{\partial N}{\partial t}\right) / N \sigma_a$$

The whole calculation is carried-out for a reactor operating time

$T_1$ days. The poison cross-section is calculated from the equation 5.1 as

$$\sum_{ap} = \frac{D\dfrac{\partial^2\phi}{\partial x^2}}{\phi} + (\nu\sigma_f - \sigma_a)\,N$$

The operating time is increased every time by a certain number of days and the whole calculation is repeated to calculate the poison cross-section for each operating time, until the poison cross-section reduces to a preset value. Since this preset value was not available, the reactor was operated for a maximum of 300 days and the final burnup data was obtained at T = 300 days.

5.5 Results of the depletion problem as discussed in this chapter for a slab reactor of width 200 cm flueld with $U_{235}$ ( $\frac{N_{25}}{N_{28}}$ = .4) with initial multiplication factor of 1.032 are discussed in this section.

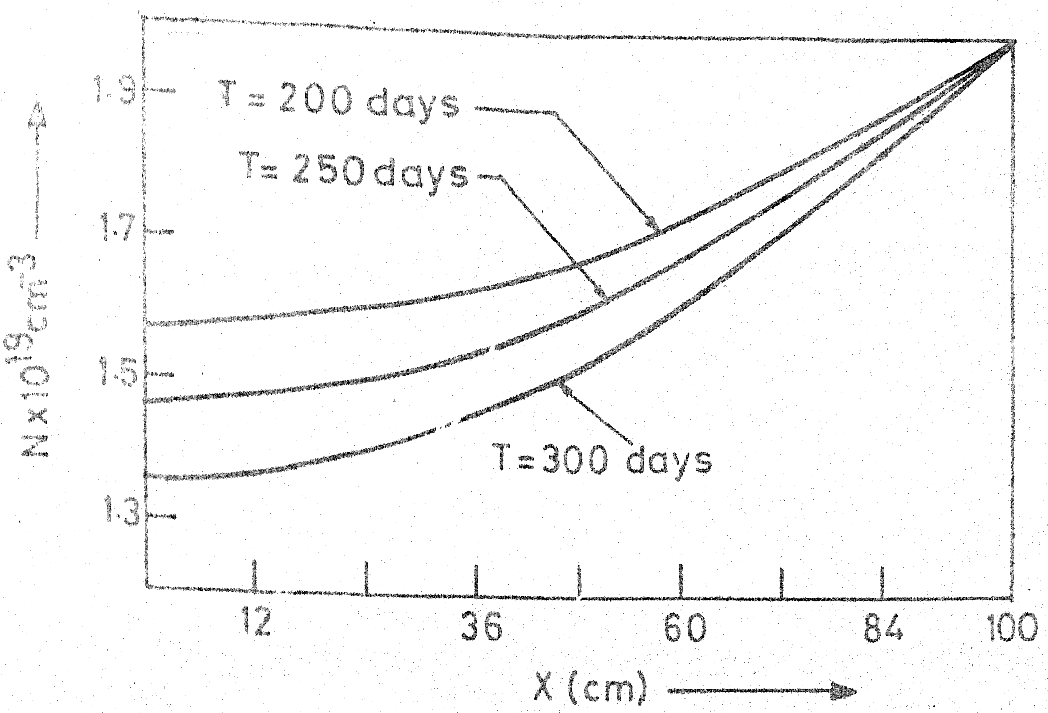
Input parameters are same as in chapter 3. The only difference is that the trial functions in this chapter assume linear polynomial in time as against a second degree polynomial in chapter 3. The power is assumed constant as opposed to a polynomial in time in chapter 3. The table 5.1 shows the neutron flux and fuel density distribution over the space for different operating times T. They are plotted in figure 5.1. The figure 5.2 shows the variation of poison cross-section with reactor operating times T. As noticed in earlier

chapters, the poison cross-section steadily goes down with time. The reason is that $U_{235}$ is the only fissile material and it is assumed that there is no buildup of $PU_{239}$ from $U_{238}$. From figure 5.2 it is seen that $\Sigma_{ap}$ is smaller in this case compared to $\Sigma_{ap}$ obtained in chapter 2. It is therefore evident from equation 5.1 that the fissile number density N is less compared to that obtained in chapter 2. Since N has gone down, the flux $\phi$ has gone up to maintain a constant power as seen from equation 5.3. But since the present problem deals with an optimal control formulation, the results cannot be directly compared with the results of chapter 2.

| x cm | $\phi \times 10^{-13}$ T = 200 days | $\phi \times 10^{-13}$ T = 250 days | $\phi \times 10^{-13}$ T = 300 days | $N \times 10^{-19}$ T = 200 days | $N \times 10^{-19}$ T = 250 days | $N \times 10^{-19}$ T = 300 days |
|---|---|---|---|---|---|---|
| 0 | 7.91 | 8.52 | 9.20 | 1.57 | 1.46 | 1.35 |
| 10 | 7.81 | 8.41 | 8.95 | 1.58 | 1.47 | 1.36 |
| 20 | 7.52 | 8.03 | 8.51 | 1.59 | 1.49 | 1.38 |
| 30 | 6.79 | 7.46 | 7.80 | 1.62 | 1.52 | 1.42 |
| 40 | 6.10 | 6.52 | 6.89 | 1.64 | 1.56 | 1.47 |
| 50 | 5.25 | 5.57 | 5.84 | 1.70 | 1.62 | 1.54 |
| 60 | 4.19 | 4.38 | 4.56 | 1.75 | 1.69 | 1.62 |
| 70 | 3.20 | 3.36 | 3.48 | 1.79 | 1.75 | 1.71 |
| 80 | 2.12 | 2.18 | 2.30 | 1.86 | 1.83 | 1.79 |
| 90 | 1.04 | 1.08 | 1.16 | 1.93 | 1.92 | 1.89 |
| 100 | 0.00 | 0.00 | 0.00 | 1.99 | 1.99 | 1.99 |

TABLE 5.1

FUEL AND FLUX DISTRIBUTION


1.9
1.7
1.5
1.3
T = 200 days
T= 250 days
T=300 days
N x 10^19 cm^-3
12
36
60
84
100
X (cm)

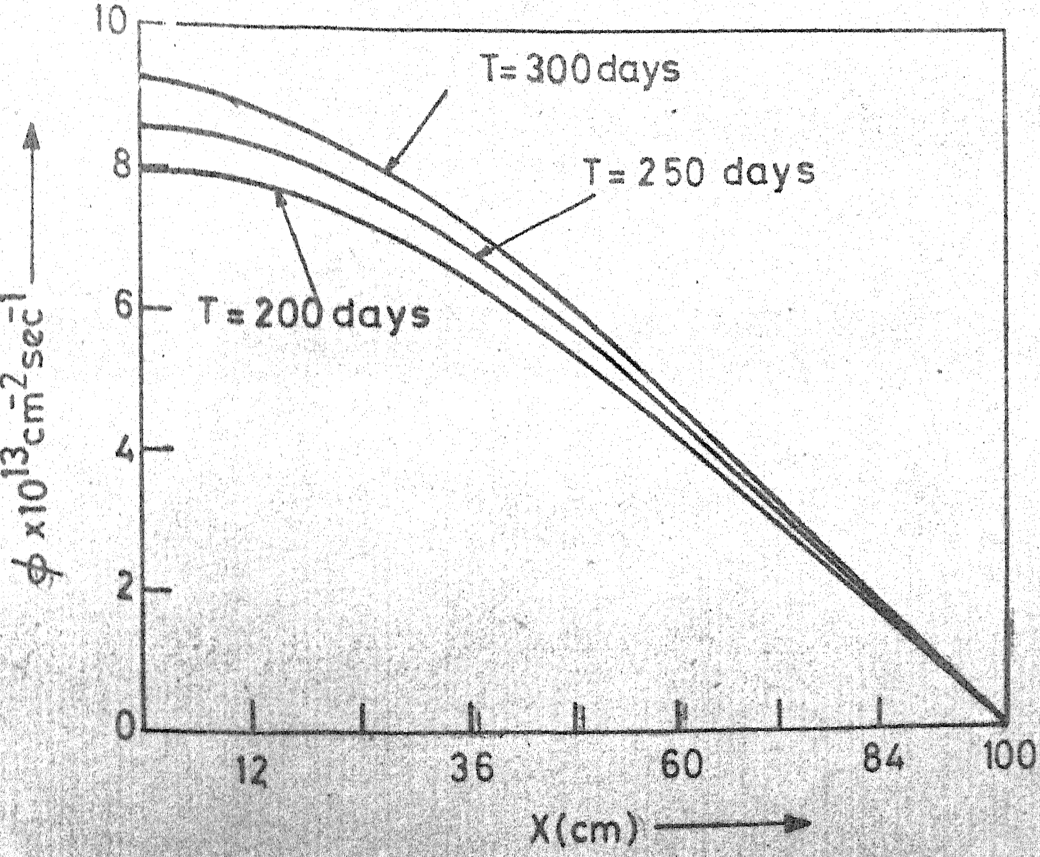
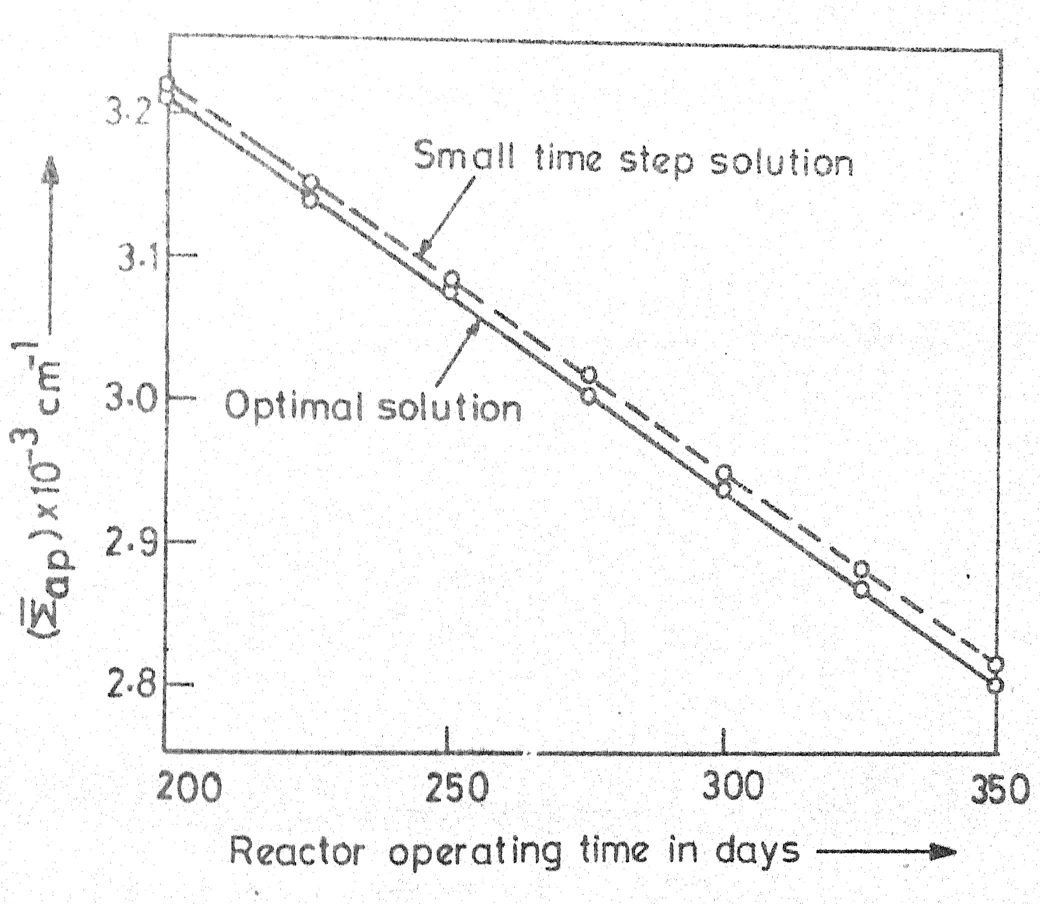

10
8
6
4
2
0
T= 300 days
T= 250 days
T= 200 days
φ x10^13 cm^-2 sec^-1
12
36
60
84
100
X(cm)

FIG. 5.1



3.2
3.1
3.0
2.9
2.8
$(\bar{\Sigma}_{ap}) \times 10^{-3}\ cm^{-1}$
Small time step solution
Optimal solution
200
250
300
350
Reactor operating time in days


FIG. 5.2

# CHAPTER - 6

From the analysis carried-out in the present work it is evident that the methods adopted here can tackle complicated non-linear initial or boundary value problem in the form of partial differential equations with constraint equation.

But the depletion problem treated in the present study is greatly simplified. Some important factors such as conversion of $U^{238}$ to $Pu^{239}$, build-up of fission products, presence of a reflector about the core, possibility of multiregion core are not considered here. So these refinements may be incorporated in the depletion problem for better understanding of the problem.

In many nuclear reactors plutonium production at the beginning of life more then offsets $U^{235}$ consumption causing an initial increase in reactivity. The poison level should therefore increase to a maximum value before beginning to decrease.

Taking plutonium production into consideration the governing equations for the depletion problem are changed as,

Neutron balance equation :

$$D \frac{\partial^2 \phi}{\partial x^2} + (\nu \sigma_f - \sigma_a) N \phi - \Sigma_{ap} \phi + (\nu_9 \sigma_{f9} - \sigma_{a9}) N_9 \phi = 0 \qquad (6.1)$$

where

$N_9$ = number density of plutonium

$\sigma_{f9}$ = fission cross-section of plutonium

$\nu_9$ = average number of neutrons per fission of plutonium.

Dynamic equations :

The rate equation for uranium is unchanged

$$\frac{\partial N}{\partial t} = - \sigma_a \phi N \qquad 6.2$$

The rate equation for plutonium is given by

$$\frac{\partial N_9}{\partial t} = - \sigma_{a9} \phi N_9 + \sigma_{c8} N_8 \phi \qquad 6.3$$

where the first term is due to neutron absorption (fission plus capture) in $Pu_{239}$ and the second term is due to plutonium formation from thermal neutron capture in $U_{238}$.

$\sigma_{a9}$ = absorption cross-section pf $Pu_{239}$

$\sigma_{c8}$ = capture cross-section of $U_{238}$

$N_8$ = number density of $U_{238}$.

As a first approximation, the concentration of $U_{238}$, which changes by a few per cent only during normal reactor operating time, may be taken as constant.

The constraint equation :

$$\sigma_f \int_0^a dx\, N \phi + \sigma_{f9} \int_0^a dx\, N_9 \phi = \bar{P}(t) \qquad 6.4$$

where, $\bar{P}(t) = P(t)/G$.

All other symbols used here have already been defined. For smooth spatial power density the performance index may be written as

$$P.I. = \int_0^T \int_0^a [\alpha_1(\sigma_f N\phi + \sigma_{f9} N_9 \phi - \frac{\bar{P}}{a})^2 + \alpha_2(\frac{\partial}{\partial x}(\sigma_f N\phi + \sigma_{f9} N_9 \phi - \frac{\bar{P}}{a}))^2]\, dx\, dt \quad \ldots \; 6.5$$

The trial function for N is given as

$$N = N_o + \sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} t^m \qquad 6.6$$

The trial function for $N_9$ may be taken as

$$N_9 = c \sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} t^m \qquad 6.7$$

where c is a constant number. c can be determined from small time step calculation and is assumed to be constant for the normal reactor operation time.

By using 6.2 and 6.3 in 6.5, the performance index changes to,

$$P.I. = \int_0^T \int_0^a [\alpha_1(-\frac{\sigma_f}{\sigma_a}\frac{\partial N}{\partial t} - \frac{\sigma_{f9}}{\sigma_{a9}}\frac{\partial N_9}{\partial t} + \sigma_{f9} \cdot \frac{\sigma_{c8}}{\sigma_{a9}} N_8 \phi - \frac{\bar{P}}{a})^2$$

$$+ \sigma_2(\frac{\partial}{\partial x}(-\frac{\sigma_f}{\sigma_a}\frac{\partial N}{\partial t} - \frac{\sigma_{f9}}{\sigma_{a9}}\frac{\partial N_9}{\partial t} + \sigma_{f9} \cdot \frac{\sigma_{c8}}{a9} N_8 - \frac{\bar{P}}{a}))^2\, dx\, dt \quad \ldots \; 6.8$$

Now, from 6.2

$$\phi = -\frac{\partial N}{\partial t} / \sigma_a N$$

$$\text{or, } \phi = \frac{-\{\sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} \cdot m\, t^{m-1}\}}{\sigma_a N_o \{1 + \frac{\sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} t^m}{N_o}\}}$$

Assuming $\sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} t^m << N_o$,

$$\phi = \frac{-\{\sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} m t^{m-1}\}\{1 - \frac{\sum_{\ell=1}^{L} \sum_{m=1}^{M} b_{\ell m} \cos \frac{K_\ell \pi x}{2a} t}{N_o}\}}{\sigma_a N_o} \quad \ldots \quad 6.9$$

Using 6.9 for $\phi$ in the performance index given by 6.8 it is found that

$$\text{P.I.} = f(b_{\ell m}) \qquad \ell = 1,2,\ldots,L$$
$$m = 1,2,\ldots,M$$

Now, by the help of constraint relation 6.4, the constrained coefficients ($b_{1m}$ ; $m = 1,2,\ldots,M$) may be substituted in terms of the independent coefficients ($b_{\ell m}$ ; $\ell = 2,3,\ldots,L$ ; $m = 1,2,\ldots,M$). Thus, performance index is a function of independent variables,

$$\text{P.I.} = f(b_{\ell m}) \qquad \ell = 2,3,\ldots,L$$
$$m = 1,2,\ldots,M$$

The coefficients $b_{\ell m}$ can be determined by solving a system of nonlinear algebraic equations resulting from

$$\frac{\partial \text{ P.I.}}{\partial b_{\ell m}} = 0 \qquad \begin{array}{l} \ell = 2,3,\ldots,L \\ m = 1,2,\ldots,M \end{array} \tag{6.10}$$

Once $b_{\ell m}$ are solved, the rest part of the solution is quite easy.

For the multiregions problems, the interface boundary conditions for the neutron flux and current are to be satisfied. These may be incorporated in the performance index of the problem in the following way :

For a two region one dimensional problem let the fluxes are denoted by $\phi_1(x,t)$ and $\phi_2(x,t)$. At the interface b

$$\phi_1(b,t) = \phi_2(b,t) \tag{6.11}$$

$$D_1 \frac{\partial \phi_1}{\partial x}(x,t)\Big|_{x=b} = D_2 \frac{\partial \phi_2}{\partial x}(x,t)\Big|_{x=b} \tag{6.12}$$

$D_1$, $D_2$ being diffusion coefficients in the two regions.

These equations 6.11 and 6.12 can be taken care of by including the terms

$$\int_0^T [\phi_1(b,t) - \phi_2(b,t)]^2 \tag{6.13}$$

$$\int_0^T [D_1 \frac{\partial \phi_1(x,t)}{\partial x}\Big|_{x=b} - D_2 \frac{\partial \phi_2}{\partial x}(x,t)\Big|_{x=b}]^2 \, dt \tag{6.14}$$

to the performance index.

The reflector may be considered as an additional region and the above mentioned technique may be used or an unreflected core with reflector savings may be treated.

The methods adopted here may be extended for three dimensional physical reactor problems by assuming three dimensional trial functions for the state variables. Numerical evaluation of triple integrals will be involved.

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